

# MATH 214

## Chapter 16: Vector Calculus

### 0.1 Line Integrals

Consider a smooth plane curve  $\mathcal{C}$  in the space given by the parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad a \leq t \leq b \quad (1)$$

or  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . Then,  $\mathbf{r}'$  is continuous and  $\mathbf{r}'(t) \neq 0$ .

*Construct a uniform partition of  $[a, b]$  into  $n$  subintervals  $[t_{i-1}, t_i]$  with a point  $t_i^*$  in it. Show Fig 1 book for corresponding partition along arc length parameter  $s$ .*

### 0.2 Definition 0.1: Line integral of Scalar Function $f$ along $\mathcal{C}$

-  $f$  is defined on smooth curve  $\mathcal{C}$  given by equations (1)

Then,

$$\int_{\mathcal{C}} f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

is the line integral of  $f$  along  $\mathcal{C}$ , if this limit exists.

### 0.3 Theorem 0.1: Evaluation of Line Integral of a Scalar Function $f$ along $\mathcal{C}$

Discuss: Evaluation of arc length of curve  $\mathcal{C}$  between  $a$  and  $b$ .

The line integral of  $f$  along  $\mathcal{C}$  can be evaluated as

$$\int_{\mathcal{C}} f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

#### 0.4 Theorem 0.2: Line Integral of a Scalar Function $f$ along $\mathcal{C}$ with respect to $x$ , $y$ , and $z$

$$\begin{aligned}\int_{\mathcal{C}} f(x, y, z) dx &= \int_a^b f(x(t), y(t), z(t)) x'(t) dt \\ \int_{\mathcal{C}} f(x, y, z) dy &= \int_a^b f(x(t), y(t), z(t)) y'(t) dt \\ \int_{\mathcal{C}} f(x, y, z) dz &= \int_a^b f(x(t), y(t), z(t)) z'(t) dt\end{aligned}$$

#### 0.5 Definition 0.2: Vector Field

If  $E \subseteq \mathbb{R}^3$ , then a vector field on  $\mathbb{R}^3$  is a function  $F$  that assigns to each point  $(x, y, z)$  in  $E$  a three-dimensional vector  $\mathbf{F}(x, y, z)$ . It can be expressed as

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

#### 0.6 Definition 0.3: Flow Lines or Streamlines

The flow lines of a vector field  $\mathbf{F}$  are the curves  $\mathcal{C}$  in the space (or plane) such that the vectors in the vector field are tangents to these curves.

*Alternative Definition:*

The flow lines or streamlines of a vector field are the paths followed by particles whose velocity field is the given vector field.

More precisely, if

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

and a flow line curve  $\mathcal{C}$  has the parametric representation  $\mathbf{r}(t) = (x(t), y(t), z(t))$  then the components of  $\mathbf{r}$  satisfy the differential equation

$$\frac{dx}{dt}(t) = P(x(t), y(t), z(t)) \quad \frac{dy}{dt}(t) = Q(x(t), y(t), z(t)) \quad \frac{dz}{dt}(t) = R(x(t), y(t), z(t))$$

#### 0.7 Definition 0.4: Work Done to Move Particle with Force $\mathbf{F}$ along $\mathcal{C}$

$$W = \int_{\mathcal{C}} \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) ds = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds$$

where,  $\mathbf{T}(x, y, z)$  is the unit tangent vector at the point  $(x, y, z)$  on  $\mathcal{C}$ .

**0.8 Theorem 0.3: Work Done to Move Particle with Force  $\mathbf{F}$  along  $\mathcal{C}$ , Using a parametric Representation of  $\mathcal{C}$**

$$W = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

*Show Why?*

**0.9 Definition 0.5: Line Integral of a Vector Field  $\mathbf{F}$  along  $\mathcal{C}$**

- $\mathbf{F}$  is a continuous vector field defined on  $\mathcal{C}$
- $\mathcal{C}$  is a smooth curve given by  $\mathbf{r}(t)$ ,  $a \leq t \leq b$

Then, the line integral of the vector field  $\mathbf{F}$  along  $\mathcal{C}$  is given by

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds$$

**0.10 Alternative Representation of a Line Integral of a Vector Field  $\mathbf{F}$  along  $\mathcal{C}$**

- If  $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$
- $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

Then,

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_{\mathcal{C}} P(x, y, z) dx + \int_{\mathcal{C}} Q(x, y, z) dy + \int_{\mathcal{C}} R(x, y, z) dz$$

*Show Why?*

**0.11 Theorem 1: Integration of Conservative Vector Fields**

- $\mathcal{C}$  is a smooth curve given by  $\mathbf{r}(t)$  where  $a \leq t \leq b$ ,
- $\mathbf{r}(a) = \langle x_1, y_1, z_1 \rangle$ , and  $\mathbf{r}(b) = \langle x_2, y_2, z_2 \rangle$
- $f$  is defined on a domain  $D$  containing  $\mathcal{C}$ ,
- $f$  is differentiable and its gradient vector  $\nabla f$  is continuous on  $\mathcal{C}$ . Then,

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

*Work on proof.*

## 0.12 Alternative for Theorem 1: Integration of Conservative Vector Fields

- $\mathbf{F}$  is continuous on  $D \subseteq \mathbb{R}^3$ ,
  - $D$  contains a smooth curve  $C$  given by  $\mathbf{r}(t)$  where  $a \leq t \leq b$ ,  
•  $\mathbf{r}(a) = \langle x_1, y_1, z_1 \rangle$ , and  $\mathbf{r}(b) = \langle x_2, y_2, z_2 \rangle$
  - $\mathbf{F}$  is a conservative vector field in the domain  $D$ . It means there is  $f$  such that  $\mathbf{F} = \nabla f$ .
- Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

*Work on proof and discuss examples*

## 0.13 Definition 1: Independence of Path

- $\mathbf{F}$  continuous vector field on  $D$ .
  - $C_1$  and  $C_2$  two curves or paths contained in  $D$ .
  - $C_1$  and  $C_2$  have the same initial and terminal point.
- Then, the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is **independent of path** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

## 0.14 Theorem 2:

$\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$  if and only if  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for every piecewise-smooth closed path  $C$  in  $D$ .

Discuss proof.

*A closed path is one for which its terminal point coincides with its initial point.*

## 0.15 Corollary:

If  $\mathbf{F}$  is a conservative vector field defined on  $D$  then, the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$ .

*Is the reciprocal statement true?*

### 0.16 Definition 2: Open and Connected Sets

- $D$  in  $\mathbb{R}^{2,3}$  is open if for every point  $P$  in  $D$  there is a disk (or ball) with center on  $P$  that is contained in  $D$ .
- $D$  is a connected set if any two points in  $D$  can be joined by a curve contained in  $D$ .

### 0.17 Theorem 3

- $\mathbf{F}$  is a continuous vector field on an open connected region  $D$ .
- $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$ .

Then  $\mathbf{F}$  is a conservative vector field on  $D$  (there is  $f$  such that  $\nabla f = \mathbf{F}$ )

### 0.18 Theorem 4: Property of a Conservative Vector Field.

- $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  is a conservative vector field on  $D$ .
- $P$  and  $Q$  have continuous first-order partial derivatives on the domain  $D$ .

Then, for all  $(x, y)$  in  $D$

$$P_y(x, y) = Q_x(x, y)$$

*work on proof*

### 0.19 Theorem 5: Easy Way to Identify a Conservative Vector Field.

- $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  is a vector field defined on an open and simply connected region  $D$ .
- $P$  and  $Q$  have continuous first-order partial derivatives on the domain  $D$ .

If  $P_y(x, y) = Q_x(x, y)$  for all  $(x, y)$  in  $D$ , then,  $\mathbf{F}$  is conservative.

*How can we find the potential  $f$  corresponding to a conservative vector field?  
Show examples*

## 0.20 Theorem 6: Green's Theorem

- $C$  is positively oriented, piecewise smooth, simple closed curve in the plane and is the boundary of a region  $D$ .
- $P$  and  $Q$  have continuous first order partial derivatives on an open region that contains  $D$ . Then,

$$\int_C P dx + Q dy = \int \int_D (Q_x - P_y) dA$$

## 0.21 Corollary 1: Green's Theorem for Regions with a Hole

- If  $D$  is a region enclosed by two simple and piecewise smooth curves  $C_1$  and  $C_2$
- $C_2$  is contained in the region enclosed by  $C_1$ .
- $C_1$  is positively oriented (counterclockwise for this case), and  $C_2$  has the same orientation as  $C_1$
- $P$  and  $Q$  have continuous first order partial derivatives on an open region that contains  $D$ .

Then,

$$\int \int_D (Q_x - P_y) dA = \int_{C_1} P dx + Q dy - \int_{C_2} P dx + Q dy$$

## 0.22 Corollary 2: Line Integral Over Complex Curves

- If  $D$  is a region enclosed by two simple piecewise and smooth curves  $C_1$  and  $C_2$
- $C_2$  is contained in the region enclosed by  $C_1$ .
- $C_1$  is positively oriented (counterclockwise for this case), and  $C_2$  has the same orientation as  $C_1$
- $P$  and  $Q$  have continuous first order partial derivatives on an open region that contains  $D$ .
- $P_y(x, y) = Q_x(x, y)$  on the open region that contains  $D$ . Then,

$$\int_{C_1} P dx + Q dy = \int_{C_2} P dx + Q dy$$

### 0.23 Definition 3: Parametric Surfaces

- Consider a region  $D$  in the  $uv$ -plane
  - and functions  $x = x(u, v)$ ,  $y = y(u, v)$ , and  $z = z(u, v)$  defined on  $D$ .
- Then, the vector-valued function defined on  $D$  as

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \quad (2)$$

is called a parametric equation of the parameters  $u$  and  $v$  and the set of points  $\{(x(u, v), y(u, v), z(u, v)) : (u, v) \in D\}$  is called a parametric surface  $S$  corresponding to the vector function  $\mathbf{r}$ .

### 0.24 Definition 4: Normal Vector to a Parametric Surface

- Consider a parametric surface  $S$  given by (2) defined on  $D$ .
- and  $x = x(u, v)$ ,  $y = y(u, v)$ , and  $z = z(u, v)$  have partial derivatives on  $D$  and

$$\mathbf{r}_u \times \mathbf{r}_v = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \neq 0, \quad \text{on } D \quad (3)$$

Then, the surface  $S$  is smooth and the normal vector to its tangent plane at  $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$  is defined as  $\mathbf{r}_u \times \mathbf{r}_v(u_0, v_0)$

### 0.25 Definition 5: Surface Area of S

- If  $S$  is smooth parametric surface given by (2).
- Then, the surface area of  $S$  is defined as

$$A(S) = \int \int_D |\mathbf{r}_u \times \mathbf{r}_v| dA \quad (4)$$

### 0.26 Theorem 7: Surface Area of the Graph of $z = f(x, y)$

- For  $z = f(x, y)$ , we can define the corresponding parametric equations as  $x = x$ ,  $y = y$ , and  $z = f(x, y)$
- If  $f$  has continuous partial derivatives, then the surface area of  $S$  is given by

$$A(S) = \int \int_D \sqrt{1 + z_x^2 + z_y^2} dA.$$

### 0.27 Definition 6: Surface Integral of a Scalar Function $f$

- If  $f$  is continuous on a region  $R$  containing the surface  $S$ .
  - $S$  is a smooth surface and  $\mathbf{r}_u$  and  $\mathbf{r}_v$  are nonparallel in  $D$ .
- Then, the surface integral of  $f$  over the surface  $S$  is defined as

$$\int \int_S f(x, y, z) dS = \int \int_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA \quad (5)$$

In particular, if  $S$  is defined as  $z = g(x, y)$ , then

$$\int \int_S f(x, y, z) dS = \int \int_D f(x, y, g(x, y)) \sqrt{z_x^2 + z_y^2 + 1} dA \quad (6)$$

*Work on Example 3 of the book*

### 0.28 Definition 7: Surface Orientation

A surface  $S$  for which there is a continuously varying unit vector  $\mathbf{n}$  is called an oriented surface. The vector  $\mathbf{n}$  provides an orientation.

For a closed surface  $S$  enclosing a solid  $E$ , positive orientation is defined as the one for which the normal vectors point out from  $E$ .

### 0.29 Definition 8: Surface Integral of Vector Fields

- $\mathbf{F}$  is a continuous vector field defined on oriented surface  $S$  with unit normal vector  $\mathbf{n}$ , then the surface integral of  $\mathbf{F}$  over  $S$  is defined as

$$\int \int_S \mathbf{F} \cdot \mathbf{S} = \int \int_S \mathbf{F} \cdot \mathbf{n} dS \quad (7)$$

It is also called flux of  $\mathbf{F}$  across  $S$ .

If  $S$  is given by the parametric function  $\mathbf{r}(u, v)$  then,

$$\int \int_S \mathbf{F} \cdot \mathbf{S} = \int \int_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA \quad (8)$$

*Show this and work on Example 5 if time permits*

### 0.30 Theorem 8: Stoke's Theorem

- $S$  is an oriented piecewise-smooth surface, bounded by a simple, closed, piecewise-smooth boundary curve  $C$  with positive orientation.
- $F$  is a vector field whose components have continuous partial derivatives on a region  $R$  that contains the surface  $S$ . Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \quad (9)$$

*Work on Example 1*

Green's theorem is special case of Stoke's Theorem. In fact,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int \int_S \text{curl } \mathbf{F} \cdot \mathbf{k} \, dA \quad (10)$$

### 0.31 Theorem 9: The Divergence or Gauss Theorem

- $E$  is a simple solid region with surface boundary  $S$ .
- $S$  has positive (outward) orientation.
- $F$  is a vector field whose component functions have continuous partial derivatives on an open region  $R$  containing  $E$ . Then,

$$\int \int \int_E \text{div } \mathbf{F} \, dV = \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_S \mathbf{F} \cdot \mathbf{n} \, dS, \quad (11)$$

where  $\mathbf{n}$  is the unit outward normal vector to  $S$

*Work on Example 2*